

# Non parametric classification

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## Probabilistic classification methods recap

The application of probabilistic classifier requires that the (at least approximate) knowledge of a suitable distribution is derived from the training set

- the class conditional distribution  $p(C_k|\mathbf{x})$  for each class  $C_k$  in the discriminative case, where an item  $\mathbf{x}$  shall be assigned to  $C_i$  if

$$i = \operatorname{argmax}_k p(C_k|\mathbf{x})$$

- the class conditional distribution  $p(\mathbf{x}|C_k)$  (and the prior distribution  $p(C_k)$ ) for each class  $C_k$  in the generative (bayesian) case, where an item  $\mathbf{x}$  shall be assigned to  $C_i$  if

$$i = \operatorname{argmax}_k p(\mathbf{x}|C_k)p(C_k)$$

## Parametric approach

The **type** of probability distribution is assumed to be known: the value of a suitable set of coefficients must be derived. For example,

- $p(C_k|\mathbf{x})$  is assumed to be of the type  $\frac{e^{\mathbf{w}_k^T \bar{\mathbf{x}}}}{\sum_i e^{\mathbf{w}_i^T \bar{\mathbf{x}}}}$  in the case of softmax (a discriminative method)
- $p(\mathbf{x}|C_k)$  is assumed to be of the type  $\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \Sigma_k)$  in the case of gaussian discriminant analysis (a generative method)

In both case, an estimate of parameter values (either  $\mathbf{w}_k$  or  $\boldsymbol{\theta}_k$ ) is performed for all classes. Different approaches to parameter estimation:

### Maximum likelihood :

- In the discriminative case, the likelihood of the target is considered  $\mathbf{w}^{ML} = \operatorname{argmax}_{\mathbf{w}} p(\mathbf{t}|\mathbf{X}, \mathbf{w})$ : prediction is performed as  $\operatorname{argmax}_k p(C_k|\mathbf{x}, \mathbf{w}^{ML})$
- In the generative case, for each class  $C_k$ , the likelihood of the subset  $\mathbf{X}_k$  of items belonging the class is instead maximized, that is  $\boldsymbol{\theta}_k^{ML} = \operatorname{argmax}_{\boldsymbol{\theta}} p(\mathbf{X}_k|\boldsymbol{\theta}_k)$ : prediction is performed as  $\operatorname{argmax}_k p(\mathbf{x}|\boldsymbol{\theta}_k^{ML})p(C_k)$

**Maximum a posteriori** : Similar to the previous one:

- In the discriminative case, the posterior of the parameters wrt to training set  $\mathbf{w}^{MAP} = \underset{\mathbf{w}}{\operatorname{argmax}} p(\mathbf{w}|\mathbf{X}, \mathbf{t})$ : prediction is performed as  $\underset{k}{\operatorname{argmax}} p(C_k|\mathbf{x}, \mathbf{w}^{MAP})$
- In the generative case, for each class  $C_k$ , the posterior of the parameters wrt the items in the class  $\boldsymbol{\theta}_k^{MAP} = \underset{\boldsymbol{\theta}_k}{\operatorname{argmax}} p(\boldsymbol{\theta}_k|\mathbf{X}_k)$  is maximized: prediction is performed as  $\underset{k}{\operatorname{argmax}} p(\mathbf{x}|\boldsymbol{\theta}_k^{MAP})p(C_k)$

**Bayesian estimate** : This approach directly express the predictive distribution as

$$p(C_k|\mathbf{x}, \mathbf{X}, \mathbf{t}) = \int_{\mathbf{w}} p(C_k|\mathbf{x}, \mathbf{w})p(\mathbf{w}|\mathbf{X}, \mathbf{t})d\mathbf{w}$$

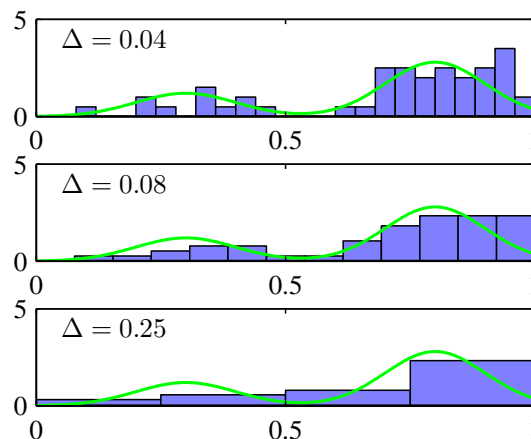
No knowledge whatsoever of the probabilities is assumed.

- The class distributions  $p(\mathbf{x}|C_i)$  are directly from data.
- In previous cases, use of (parametric) models for a synthetic description of data in  $\mathbf{X}, \mathbf{t}$
- In this case, no models (and parameters): training set items explicitly appear in class distribution estimates.
- Denoted as **non parametric** models: indeed, an unbounded number of parameters is used

## Histograms

- Elementary type of non parametric estimate
- Domain partitioned into  $m$   $d$ -dimensional intervals (**bins**)
- The probability  $P_{\mathbf{x}}$  that an item belongs to the bin containing item  $\mathbf{x}$  is estimated as  $\frac{n(\mathbf{x})}{n}$ , where  $n(\mathbf{x})$  is the number of element in that bin
- The probability density in the interval corresponding to the bin containing  $\mathbf{x}$  is then estimated as the ratio between the above probability and the interval width  $\Delta(\mathbf{x})$  (typically, a constant  $\Delta$ )

$$p_H(\mathbf{x}) = \frac{\frac{n(\mathbf{x})}{N}}{\Delta(\mathbf{x})} = \frac{n(\mathbf{x})}{N\Delta(\mathbf{x})}$$



## Kernel density estimators

- Probability that an item is in region  $\mathcal{R}(\mathbf{x})$ , containing  $\mathbf{x}$

$$P_{\mathbf{x}} = \int_{\mathcal{R}(\mathbf{x})} p(\mathbf{z}) d\mathbf{z}$$

- Given  $n$  items  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ , the probability that  $k$  among them are in  $\mathcal{R}(\mathbf{x})$  is given by the binomial distribution

$$p(k) = \binom{n}{k} P_{\mathbf{x}}^k (1 - P_{\mathbf{x}})^{n-k} = \frac{n!}{k!(n-k)!} P_{\mathbf{x}}^k (1 - P_{\mathbf{x}})^{n-k}$$

- Since  $E[k] = nP_{\mathbf{x}}$  and  $\sigma_k^2 = nP_{\mathbf{x}}(1 - P_{\mathbf{x}})$ , by the binomial distribution properties, we have that, for what concerns the ratio  $r = \frac{k}{n}$ ,

$$E[r] = \frac{1}{n} E[k] = P_{\mathbf{x}} \quad \sigma_r^2 = \frac{1}{n^2} \sigma_k^2 = \frac{P_{\mathbf{x}}(1 - P_{\mathbf{x}})}{n}$$

- $P_{\mathbf{x}}$  is the expected fraction of items in  $\mathcal{R}(\mathbf{x})$ , and the ratio  $r$  is an estimate. As  $n \rightarrow \infty$  variance decreases and  $r$  tends to  $E[r] = P_{\mathbf{x}}$ , we assume

$$r = \frac{k}{n} \simeq P_{\mathbf{x}}$$

## Nonparametric estimates

- Let the volume of  $\mathcal{R}(\mathbf{x})$  be sufficiently small. Then, the density  $p(\mathbf{x})$  is almost constant in the region and

$$P_{\mathbf{x}} = \int_{\mathcal{R}(\mathbf{x})} p(\mathbf{z}) d\mathbf{z} \simeq p(\mathbf{x})V$$

where  $V$  is the volume of  $\mathcal{R}(\mathbf{x})$

- since  $P_{\mathbf{x}} \simeq \frac{k}{n}$ , it then derives that  $p(\mathbf{x}) \simeq \frac{k}{nV}$

## Approaches to nonparametric estimates

Two alternative ways to exploit the relation  $p(\mathbf{x}) \simeq \frac{k}{nV}$  to estimate  $p(\mathbf{x})$  for any  $\mathbf{x}$ :

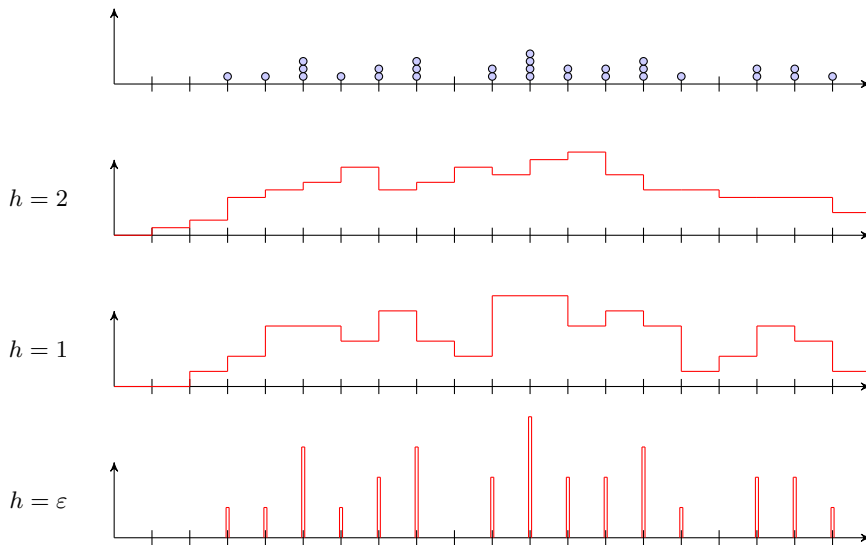
1. Fix  $V$  and derive  $k$  from data (**kernel density estimation**)
2. Fix  $k$  and derive  $V$  from data (**K-nearest neighbor**).

It can be shown that in both cases, under suitable conditions, the estimator tends to the true density  $p(\mathbf{x})$  as  $n \rightarrow \infty$ .

## Kernel density estimation: Parzen windows

- Region associated to a point  $\mathbf{x}$ : hypercube with edge length  $h$  (and volume  $h^d$ ) centered on  $\mathbf{x}$ .
- Kernel function  $k(\mathbf{z})$  (**Parzen window**) used to count the number of items in the unit hypercube centered on the origin  $\mathbf{0}$

$$k(\mathbf{z}) = \begin{cases} 1 & |z_i| \leq 1/2 \quad i = 1, \dots, d \\ 0 & \text{otherwise} \end{cases}$$



- as a consequence,  $k\left(\frac{\mathbf{x} - \mathbf{x}'}{h}\right) = 1$  iff  $\mathbf{x}'$  is in the hypercube of edge length  $h$  centered on  $\mathbf{x}$
- the number of items in the hypercube is then

$$K = \sum_{i=1}^n k\left(\frac{\mathbf{x} - \mathbf{x}_i}{h}\right)$$

- The estimated density is

$$p_n(\mathbf{x}) = \frac{1}{nV} \sum_{i=1}^n k\left(\frac{\mathbf{x} - \mathbf{x}_i}{h}\right) = \frac{1}{nh^d} \sum_{i=1}^n k\left(\frac{\mathbf{x} - \mathbf{x}_i}{h}\right)$$

- Since

$$k(\mathbf{z}) \geq 0 \quad \text{and} \quad \int k(\mathbf{z}) d\mathbf{z} = 1$$

it derives

$$k\left(\frac{\mathbf{x} - \mathbf{x}_i}{h}\right) \geq 0 \quad \text{and} \quad \int k\left(\frac{\mathbf{x} - \mathbf{x}_i}{h}\right) d\mathbf{x} = h^d$$

As a consequence, it results that  $p_n(\mathbf{x})$  is a probability density.

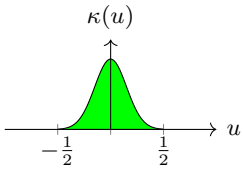
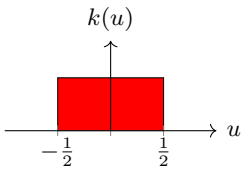
Clearly, the window size has a relevant effect on the estimate

## Kernels and smoothing

Drawbacks

1. discontinuity of the estimates
2. items in a region centered on  $\mathbf{x}$  have uniform weights: their distance from  $\mathbf{x}$  is not taken into account

Solution. Use of **smooth** kernel functions  $\kappa_h(u)$  to assign larger weights to points nearer to the origin.



Assumed characteristics of  $\kappa_h(u)$ :

$$\int \kappa_h(\mathbf{x}) d\mathbf{x} = 1$$

$$\int \mathbf{x} \kappa_h(\mathbf{x}) d\mathbf{x} = 0$$

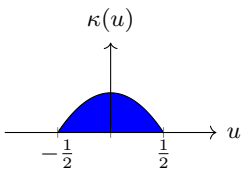
$$\int \mathbf{x}^2 \kappa_h(\mathbf{x}) d\mathbf{x} > 0$$

Usually kernels are based on smooth radial functions (functions of the distance from the origin)

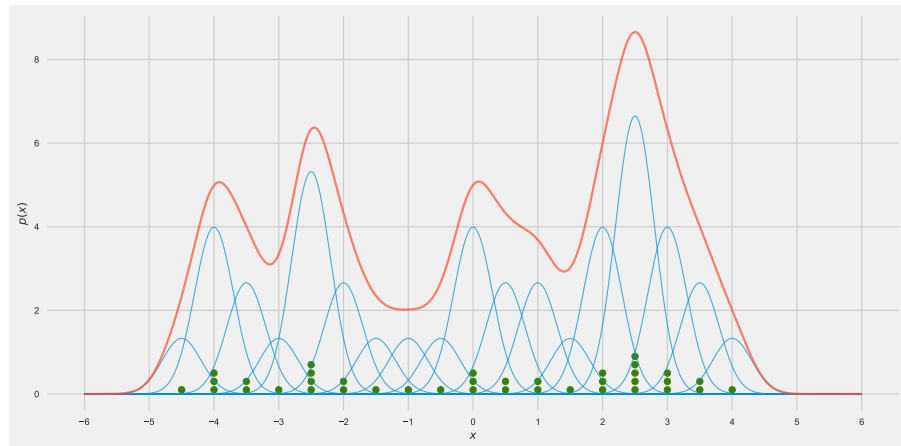
1. gaussian  $\kappa(u) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{u^2}{\sigma^2}}$ , unlimited support
2. Epanechnikov  $\kappa(u) = 3\left(\frac{1}{2} - u^2\right)$ ,  $|u| \leq \frac{1}{2}$ , limited support
3. ...

resulting estimate:

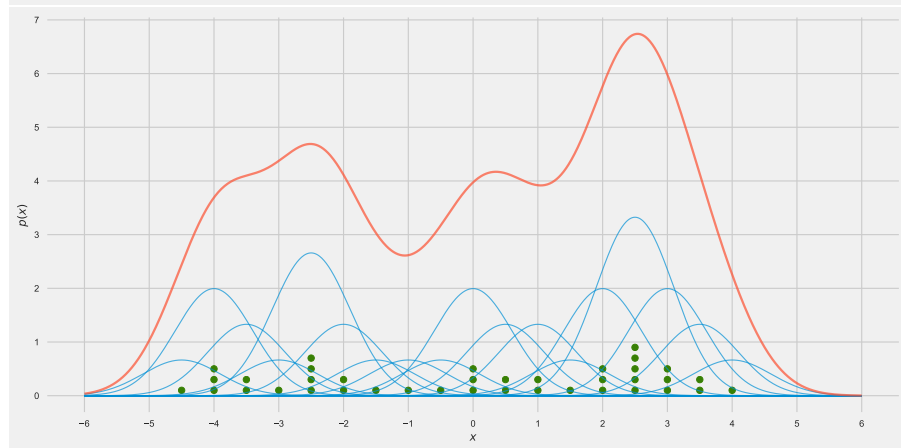
$$p(\mathbf{x}) = \frac{1}{nh} \sum_{i=1}^n \kappa\left(\frac{\mathbf{x} - \mathbf{x}_i}{h}\right) = \frac{1}{n} \sum_{i=1}^n \kappa_h(\mathbf{x} - \mathbf{x}_i)$$



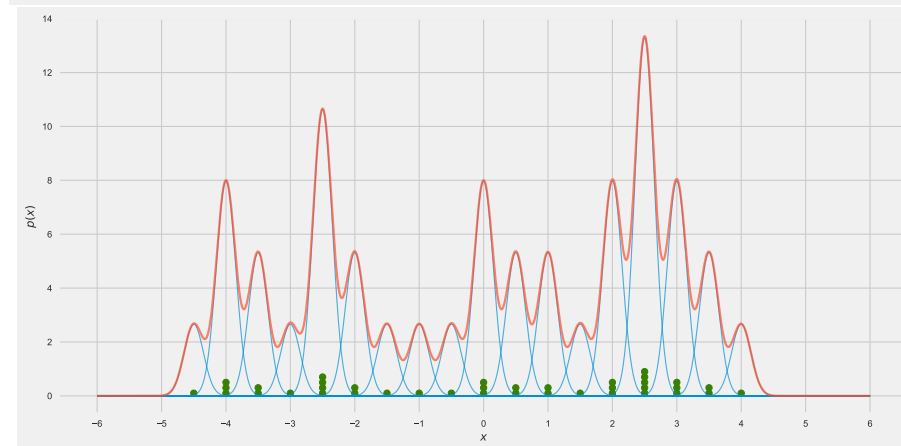
$h = 1$



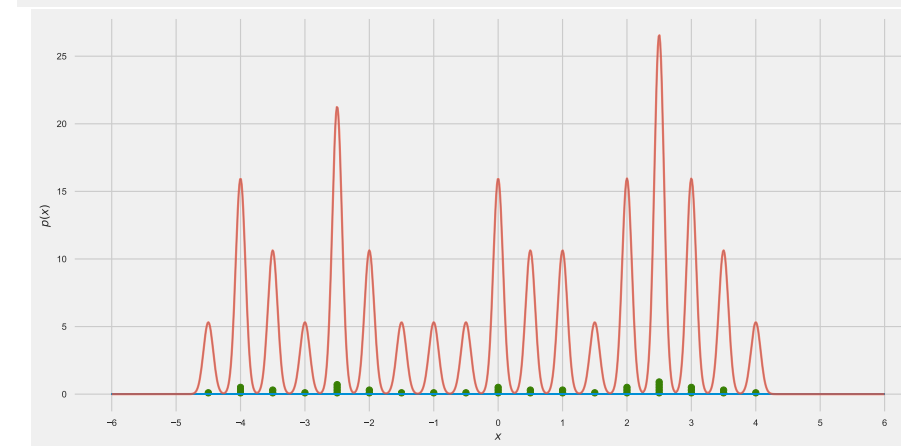
$h = 2$



$h = .5$



$h = .25$



## Parzen windows and classification

- Parzen windows provide a way to estimate  $p(\mathbf{x})$  for any  $\mathbf{x}$ , given a set of points  $\mathbf{X}$
- They can be applied to classify an item  $\mathbf{x}$  by estimating  $p(\mathbf{x}|C_k)$  for all classes, by referring to the sets  $\mathbf{X}_1, \dots, \mathbf{X}_k$  of items in the training set belonging to each class
- According to bayesian classification,  $\mathbf{x}$  is predicted to the class with index

$$\begin{aligned} \operatorname{argmax}_i p(\mathbf{x}|C_i)p(C_i) &= \operatorname{argmax}_i \frac{1}{n_i h^d} \sum_{i=1}^{n_i} k \left( \frac{\mathbf{x} - \mathbf{x}_i}{h} \right) p(C_i) = \\ &= \operatorname{argmax}_i \frac{1}{n h^d} \sum_{i=1}^{n_i} k \left( \frac{\mathbf{x} - \mathbf{x}_i}{h} \right) \\ &= \operatorname{argmax}_i \sum_{i=1}^{n_i} k \left( \frac{\mathbf{x} - \mathbf{x}_i}{h} \right) \end{aligned}$$

- that is, an item is assigned to the class with most (weighted by the kernel) points near  $\mathbf{x}$ , that is in an hypercube of edge size  $h$  with center  $\mathbf{x}$

## Density estimation through kNN

- The region around  $\mathbf{x}$  is extended to include  $k$  items
- The estimated density is

$$p(\mathbf{x}) \simeq \frac{k}{nV} = \frac{k}{n c_d r_k^d(\mathbf{x})}$$

where:

- $c_d$  is the volume of the  $d$ -dimensional sphere of unitary radius
- $r_k^d(\mathbf{x})$  is the distance from  $\mathbf{x}$  to the  $k$ -th nearest item (the radius of the smallest sphere with center  $\mathbf{x}$  containing  $k$  items)

## Classification through kNN

- To estimate  $p(C_i|\mathbf{x})$  in order to classify  $\mathbf{x}$ , let us consider a hypersphere of volume  $V$  with center  $\mathbf{x}$  containing  $k$  items from the training set
- Let  $k_i$  be the number of such items belonging to class  $C_i$ . Then, the following approximation holds:

$$p(\mathbf{x}|C_i) = \frac{k_i}{n_i V}$$

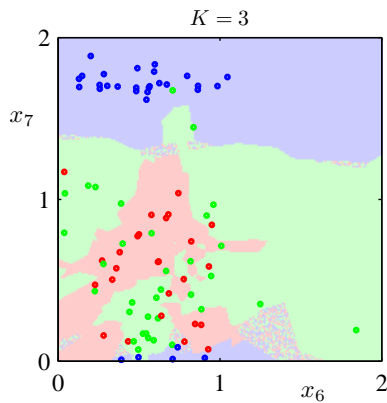
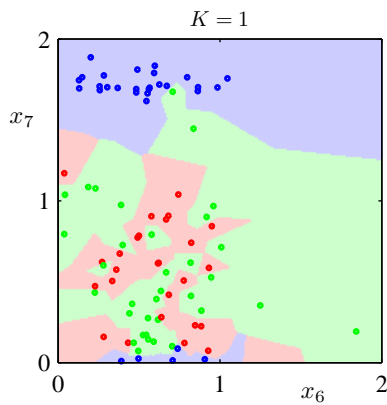
where  $n_i$  is the number of items in the training set belonging to class  $C_i$

- Similarly, for the evidence,

$$p(\mathbf{x}) = \frac{k}{nV}$$

- And, for the prior distribution,

$$p(C_i) = \frac{n_i}{n}$$



- The class posterior distribution is then

$$p(C_i|\mathbf{x}) = \frac{p(\mathbf{x}|C_i)p(C_i)}{p(\mathbf{x})} = \frac{\frac{k_i}{n_i V} \cdot \frac{n_i}{n}}{\frac{k}{nV}} = \frac{k_i}{k}$$

- Simple rule: an item is classified on the basis of similarity to near training set items
- To classify  $\mathbf{x}$ , determine the  $k$  items in the training nearest to it and assign  $\mathbf{x}$  to the majority class among them
- A metric is necessary to measure similarity.
- kNN is a simple classifier which can work quite well, provided it is given a good distance metric and has enough labeled training data: it can be shown that it can result within a factor of 2 of the best possible performance as  $n \rightarrow \infty$
- subject to the curse of dimensionality: due to the large sparseness of data at high dimensionality, items considered by kNN can be quite far away from the query point, and thus resulting in poor locality.



