Clustering

Course of Machine Learning
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Partitional clustering

Given a dataset $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, with $\mathbf{x}_i \in \mathbb{R}^d (i = 1, \ldots, n)$.

We wish to derive a set of **clusters** (i.e. a partition of $\mathbf{X}$ into subsets of "near" elements). Clusters are represented by their **prototypes** $(\mathbf{m}_1, \ldots, \mathbf{m}_k)$, with $\mathbf{m}_j \in \mathbb{R}^d, j = 1, \ldots, k$.

Representation of a clustering

1. Cluster prototypes $(\mathbf{m}_1, \ldots, \mathbf{m}_k)$, with $\mathbf{m}_j \in \mathbb{R}^d (j = 1, \ldots, k)$
2. Element assignment to clusters: for each $\mathbf{x}_i$, k binary flags $r_{ij} \in \{0, 1\}$, $j = 1, \ldots, k$. If $\mathbf{x}_i$ is assigned the t-th cluster, then $r_{it} = 1$ and $r_{ij} = 0$ for $j \neq t$

Clustering types

- Partitional clustering: Given a set of items (points) $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, we wish to partition $\mathbf{X}$ by assigning each element to one out of k clusters $C_1, \ldots, C_k$ in such a way to maximize (or minimize) a given cost J. The number k of clusters could be given or should have to be computed.
- Hierarchical clustering: Given a set of items (points) $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, we wish to derive a set of nested partitions of $\mathbf{X}$, from the partition composed by all singletons (one cluster for each node) to the one composed by a single item (the whole set).

Clustering cost

Sum of squares

Let us define the cost a clustering as follows:

$$J(R, M) = \sum_{i=1}^{k} \sum_{j=1}^{n} r_{ij} ||\mathbf{x}_j - \mathbf{m}_i||^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} r_{ij} (\mathbf{x}_j - \mathbf{m}_i)^T (\mathbf{x}_j - \mathbf{m}_i)$$

where

- $R_{ij} = r_{ij}$, where $r_{is} = 1$ and $r_{ij} = 0$ for $j \neq s$ if x_i is assigned to cluster C_s
- $M_i = \mathbf{m}_i$, $i = 1, \ldots, k$ is the prototype (centroid) of cluster C_i,

$$\mathbf{m}_i = \frac{1}{n_i} \sum_{j=1}^{n} r_{ij} \mathbf{x}_j$$

k-means clustering

Dataset $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, $\mathbf{x}_i \in \mathbb{R}^d$: we wish to derive k clusters with prototypes $\mathbf{m}_1, \ldots, \mathbf{m}_k$

Assignment of elements to cluster: for each $\mathbf{x}_i$, k binary flags r_{ij} $(j = 1, \ldots, k)$

- if $\mathbf{x}_i$ is assigned to cluster s, then $r_{is} = 1$, and $r_{ij} = 0$ for $j \neq k$

Cost: sum of the distances of each point from the prototype of the corresponding cluster

$$J(R, M) = \sum_{i=1}^{n} \sum_{j=1}^{k} r_{ij} \, ||\mathbf{x}_i - \mathbf{m}_j||^2$$

Objective: finding r_{ij} and $\mathbf{m}_j$ $(i = 1, \ldots, n, j = 1, \ldots, k)$ to minimize $J(R, M)$

Algorithm

1. Given a set of prototypes $\mathbf{m}_{ij}$, minimize wrt r_{ij} (assigning elements to clusters).

   For each $\mathbf{x}_i$, minimize $\sum_{j=1}^{k} r_{ij} \, ||x_i - \mathbf{m}_j||^2$.

   The minimum is obtained for $r_{ik} = 1$ (and $r_{ij} = 0$ for $j \neq k$), where $||\mathbf{x}_i - \mathbf{m}_k||^2$ is the minimum distance. That is, each point is assigned to the cluster of the nearest prototype.

2. Given a set of assignments r_{ij}, minimize wrt $\mathbf{m}_{ij}$ (defining new cluster prototypes)

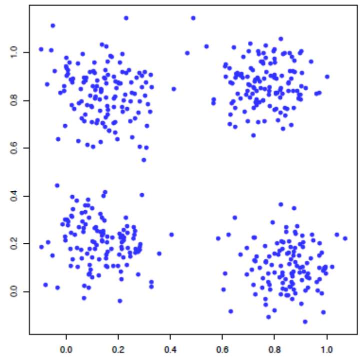
   For each $\mathbf{m}_k$, $J = \sum_{i=1}^{n} \sum_{j=1}^{k} r_{ij} \, ||\mathbf{x}_i - \mathbf{m}_j||^2$ is a quadratic function of $\mathbf{m}_k$. By setting its derivative to zero, the values of $\mathbf{m}_k$ providing its minimum are obtained

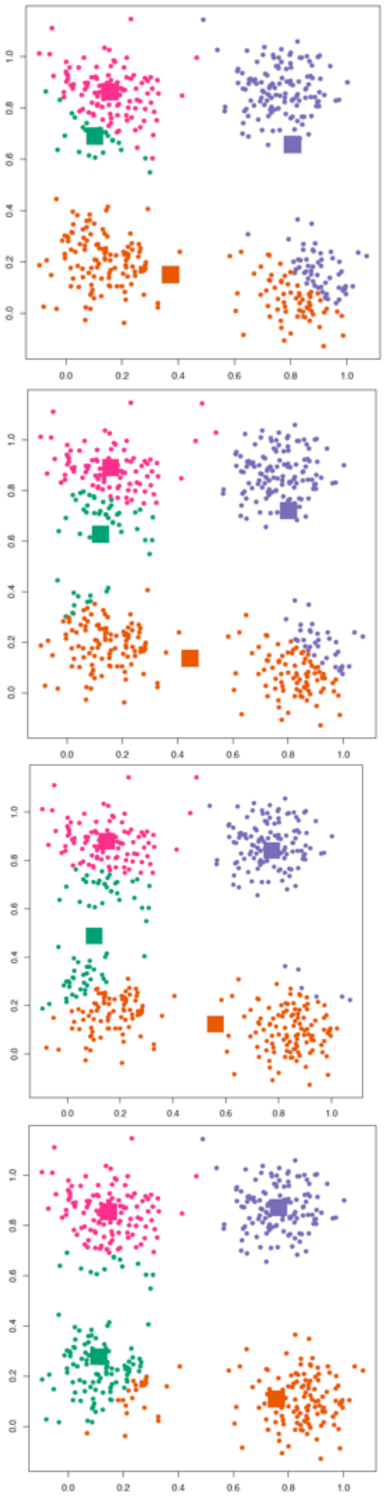
$$\frac{\partial J}{\partial \mathbf{m}_k} = 2 \sum_{i=1}^{n} r_{ik}(\mathbf{x}_i - \mathbf{m}_k) = 0 \Longrightarrow \mathbf{m}_k = \frac{\sum_{i=1}^{n} r_{ik} \mathbf{x}_i}{\sum_{i=1}^{n} r_{ik}}$$

   That is, the new prototype is the mean of the elements assigned to the cluster

At each step, J does not increase. There is a convergence to a local minimum.

Example of application of k-means





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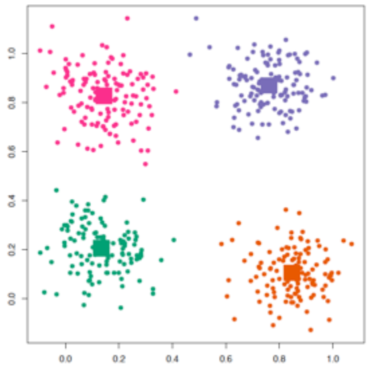
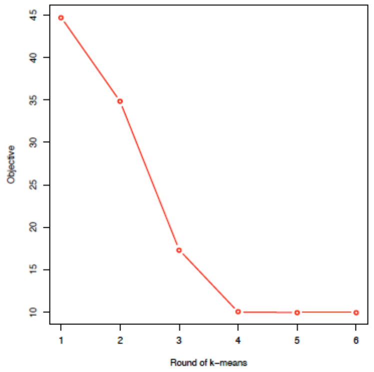




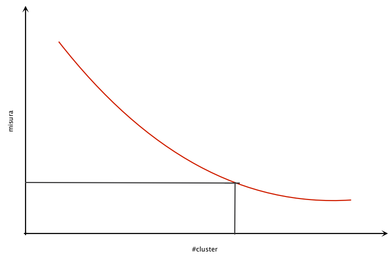
How to choose K

Cross validation

- Apply cross validation for different values of K, measuring the quality of the clustering obtained
- How to measure the quality of a clustering?
    1. mean distance of elements from the prototypes of their clusters
    2. log-likelihood of the elements wrt the resulting mixture model

Note

Measures improves as K increases (overfitting). A value such that further increases provide limited improvement should be found



Hierarchical clustering

Aim

Derivation of a binary tree. Node: cluster; arc: inclusion.

The tree specifies a set of pairwise merge of clusters.

- Aggregation, starting from n singleton clusters

- Separation, starting from a single cluster of size n

Requirements: k-means requires:

- a number K of clusters
- an initial assignment
- a distance function between elements

Hierarchical clustering requires:

- a similarity function between clusters

Algorithm

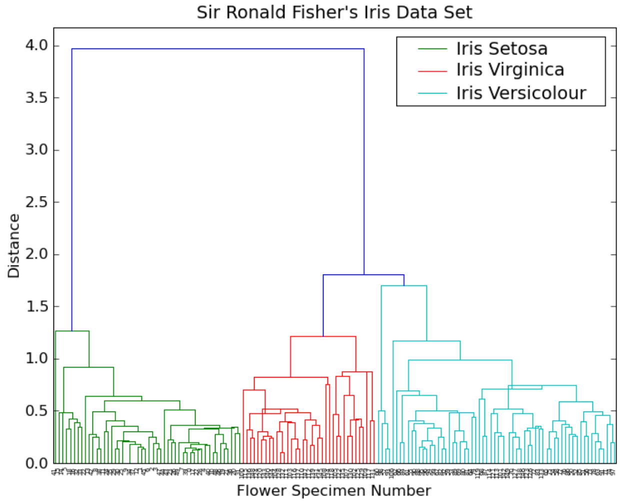
- define n clusters (singleton)
- repeat
    - compute the matrix of distances between clusters
    - merge the pair of clusters which are "nearest"
- until a single cluster has remained

Properties

- Each tree prefix is a partition of elements
- The algorithm provides a partial order of clusterings
- The best clustering has to be found
- Monotonicity: similarity between paired clusters decreases

Dendrogram

- Tree of cluster pairings
- The height of the nodes is inversely proportional to the similarity of the paired clusters



Cluster similarity

Many measures. Most frequent ones:

- Similarity between nearest nodes (**Single linkage**)

$$d_{SL}(C_1, C_2) = \min_{\mathbf{x}_1 \in C_1, \mathbf{x}_2 \in C_2} d(\mathbf{x}_i, \mathbf{x}_j)$$

- Similarity between farthest nodes (**Complete linkage**)

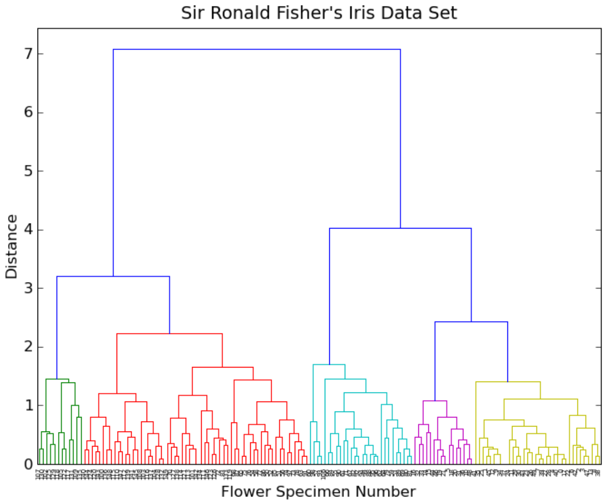
$$d_{CL}(C_1, C_2) = \max_{\mathbf{x}_1 \in C_1, \mathbf{x}_2 \in C_2} d(\mathbf{x}_i, \mathbf{x}_j)$$

- Mean similarity (**Group average**)

$$d_{GA}(C_1, C_2) = \frac{1}{|C_1| \cdot |C_2|} \sum_{\mathbf{x}_1 \in C_1} \sum_{\mathbf{x}_2 \in C_2} d(\mathbf{x}_i, \mathbf{x}_j)$$

Different measures provide different dendrograms

Dendrogram with complete linkage



Mixtures of distributions

Linear combinations of probability distributions

- Same type of distributions $q(\mathrm{x}|\theta)$
- Differ by parameter values

$$p(\mathrm{x}|\boldsymbol{\pi}, \boldsymbol{\theta}) = \sum_{k=1}^{K} \pi_k q(\mathrm{x}|\theta_k)$$

where

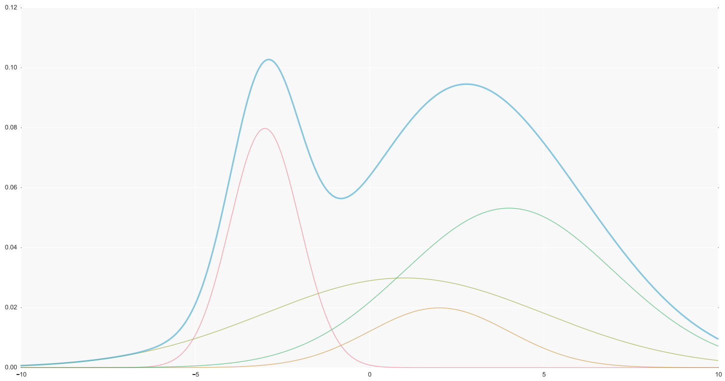
$$\boldsymbol{\pi} = (\pi_1, \ldots, \pi_K) \qquad \boldsymbol{\theta} = (\theta_1, \ldots, \theta_K)$$

Mixing coefficients

$$0 \leq \pi_k \leq 1 \quad k = 1, \ldots, K \qquad \sum_{k=1}^{K} \pi_k = 1$$

Terms π_k have the properties of probability values

Provide extensive capabilities to model complex distributions. For example, almost all continuous distributions can be modeled by the linear combination of a suitable number of gaussians.



Mixture parameters estimation

Given a dataset $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$, the parameters $\boldsymbol{\pi}, \boldsymbol{\theta}$ of a mixture can be estimated by maximum likelihood.

$$L(\boldsymbol{\theta}, \boldsymbol{\pi} | \mathrm{X}) = p(\mathrm{X} | \boldsymbol{\theta}, \boldsymbol{\pi}) = \prod_{i=1}^{n} p(\mathrm{x}_i | \boldsymbol{\theta}, \boldsymbol{\pi}) = \prod_{i=1}^{n} \sum_{k=1}^{K} \pi_k q(\mathbf{x} | \theta_k)$$

or maximum log-likelihood

$$l(\boldsymbol{\theta}, \boldsymbol{\pi} | \mathrm{X}) = \log p(\mathrm{X} | \boldsymbol{\theta}, \boldsymbol{\pi}) = \sum_{i=1}^{n} \log p(\mathrm{x}_i | \boldsymbol{\theta}, \boldsymbol{\pi}) = \sum_{i=1}^{n} \log \left(\sum_{k=1}^{K} \pi_k q(\mathbf{x}_i | \theta_k) \right)$$

Maximization is however constrained by the conditions $0 \leq \pi_i \leq 1$ for all i and $\sum_{i=1}^{K} \pi_i = 1$.

By applying the lagrangian multipliers method, we will maximize

$$\mathcal{L}(\boldsymbol{\theta}, \boldsymbol{\pi}, \lambda) = l(\boldsymbol{\theta}, \boldsymbol{\pi} | \mathrm{X}) + \lambda(1 - \sum_{i=1}^{K} \pi_i)$$

Let us first consider the derivatives with respect to the weights $\boldsymbol{\pi}$, which we set to 0

$$\frac{\partial \mathcal{L}(\boldsymbol{\theta}, \boldsymbol{\pi} | \mathrm{X})}{\partial \pi_j} = \frac{\partial l(\boldsymbol{\theta}, \boldsymbol{\pi} | \mathrm{X})}{\partial \pi_j} - \lambda = 0$$

This is equivalent to

$$\lambda = \frac{\partial l(\boldsymbol{\theta}, \boldsymbol{\pi} | \mathrm{X})}{\partial \pi_j} = \frac{\partial}{\partial \pi_j} \left[\sum_{i=1}^{n} \log \left(\sum_{k=1}^{K} \pi_k q(\mathbf{x}_i | \theta_k) \right) \right] = \sum_{i=1}^{n} \frac{\partial}{\partial \pi_j} \left[\log \left(\sum_{k=1}^{K} \pi_k q(\mathbf{x}_i | \theta_k) \right) \right]$$
$$= \sum_{i=1}^{n} \frac{q(\mathbf{x}_i | \theta_j)}{\sum_{k=1}^{K} \pi_k q(\mathbf{x}_i | \theta_k)} = \sum_{i=1}^{n} \frac{\gamma_j(\mathbf{x}_i)}{\pi_j} = \frac{1}{\pi_j} \sum_{i=1}^{n} \gamma_j(\mathbf{x}_i)$$

where,

$$\gamma_k(\mathbf{x}) = \frac{\pi_k q(\mathbf{x} | \theta_k)}{\sum_{j=1}^{K} \pi_j q(\mathbf{x} | \theta_j)}$$

By setting the derivative wrt λ to 0

$$\frac{\partial \mathcal{L}(\boldsymbol{\theta}, \boldsymbol{\pi} | \mathrm{X})}{\partial \lambda} = \frac{\partial}{\partial \lambda} \left(l(\boldsymbol{\theta}, \boldsymbol{\pi} | \mathrm{X}) + \lambda(1 - \sum_{i=1}^{K} \pi_i) \right) = 0$$

we obtain

$$\sum_{i=1}^{K} \pi_i = 1$$

As a consequence, since, as shown above,

$$\pi_j = \frac{1}{\lambda} \sum_{i=1}^{n} \gamma_j(\mathbf{x}_i)$$

it results

$$\sum_{j=1}^{K} \pi_j = \frac{1}{\lambda} \sum_{j=1}^{K} \sum_{i=1}^{n} \gamma_j(\mathbf{x}_i) = 1$$

which implies

$$\lambda = \sum_{j=1}^{K} \sum_{i=1}^{n} \gamma_j(\mathbf{x}_i) = \sum_{i=1}^{n} \sum_{j=1}^{K} \gamma_j(\mathbf{x}_i) = \sum_{i=1}^{n} \sum_{j=1}^{K} \frac{\pi_j q(\mathbf{x}_i|\theta_j)}{\sum_{k=1}^{K} \pi_k q(\mathbf{x}_i|\theta_k)} = \sum_{i=1}^{n} 1 = n$$

and, finally,

$$\pi_k = \frac{1}{n} \sum_{i=1}^{n} \gamma_k(\mathbf{x}_i)$$

For what concerns derivatives (or gradients) wrt distribution parameters $\boldsymbol{\theta}$, it results

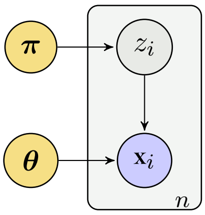
$$\begin{aligned}
\frac{\partial \mathcal{L}(\boldsymbol{\theta}, \boldsymbol{\pi}|\mathrm{X})}{\partial \theta_j} &= \frac{\partial}{\partial \theta_j} \left[\sum_{i=1}^{n} \log \left(\sum_{k=1}^{K} \pi_k q(\mathbf{x}_i|\theta_k) \right) \right] = \sum_{i=1}^{n} \frac{\partial}{\partial \theta_j} \left[\log \left(\sum_{k=1}^{K} \pi_k q(\mathbf{x}_i|\theta_k) \right) \right] \\
&= \sum_{i=1}^{n} \frac{\pi_j q(\mathbf{x}_i|\theta_j)}{\sum_{k=1}^{K} \pi_k q(\mathbf{x}_i|\theta_k)} \frac{\partial \log q(\mathbf{x}_i|\theta_j)}{\partial \theta_j} \\
&= \sum_{i=1}^{n} \gamma_j(\mathbf{x}_i) \frac{\partial \log q(\mathbf{x}_i|\theta_j)}{\partial \theta_j} = 0
\end{aligned}$$

Log likelihood maximization is intractable analytically: its solution cannot be given in closed form.

- $\boldsymbol{\pi}$ and $\boldsymbol{\theta}$ can be derived from $\gamma_k(\mathbf{x}_i)$
- Also, $\gamma_k(\mathbf{x}_i)$ can be derived from $\boldsymbol{\pi}$ e $\boldsymbol{\theta}$

Iterative techniques

- Given an estimation for $\boldsymbol{\pi}$ e $\boldsymbol{\theta}$...
- derive an estimation for $\gamma_k(\mathbf{x}_i)$, from which ...
- derive a new estimation for $\boldsymbol{\pi}$ e $\boldsymbol{\theta}$, from which ...
- derive a new estimation for $\gamma_k(\mathbf{x}_i)$ ...



Mixtures as generative processes

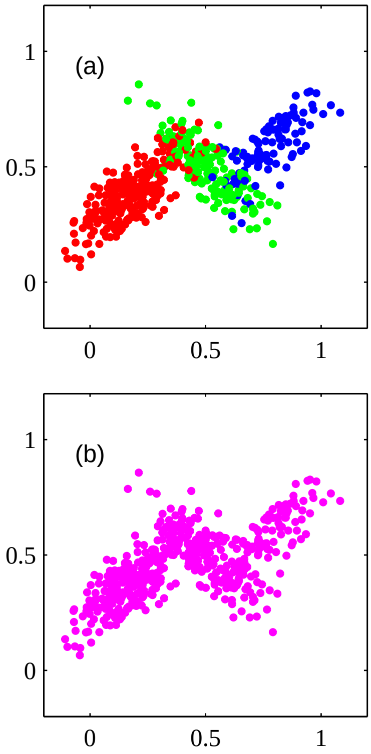
Graphical model representation of a mixture of distributions. Latent variables

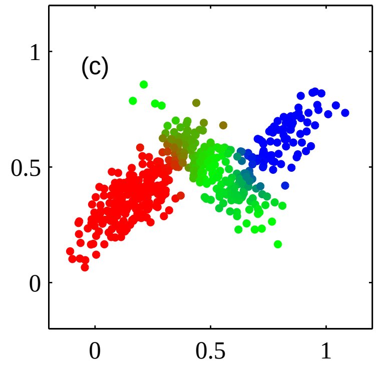
- Terms z_i are latent random variable with domain $z \in \{1, \dots, K\}$
- While $\mathbf{x}_i$ is observed, the value of z_i cannot be observed
- z_i denotes the component distribution $q(\mathbf{x}|\theta)$ responsible for the generation of $\mathbf{x}_i$

Generation process

1. Starting from the distribution $\pi_1, \dots, \pi_K$, the component distribution to apply to sample the value of $\mathbf{x}_i$ is sampled: its index is given by z_i. Hence z_i is dependent from $\boldsymbol{\pi}$
2. Let $z_i = k$: then, $\mathbf{x}_i$ is sampled from distribution $q(\mathbf{x}|\theta_k)$. That is, $\mathbf{x}_i$ is dependent from both z_i and $\boldsymbol{\theta}$ (through θ_k)

Example of generation of dataset from mixture of 3 gaussians





Distributions with latent variables

$$p(\mathbf{x}|z=k,\boldsymbol{\theta},\boldsymbol{\pi}) = p(\mathbf{x}|z=k,\boldsymbol{\theta}) = q(\mathbf{x}|\theta_k)$$

Marginalizing wrt z,

$$\begin{aligned} p(\mathbf{x}|\boldsymbol{\theta},\boldsymbol{\pi}) &= \sum_{k=1}^{K} p(\mathbf{x},z=k|\boldsymbol{\theta},\boldsymbol{\pi}) = \sum_{k=1}^{K} p(\mathbf{x}|z=k,\boldsymbol{\pi},\boldsymbol{\theta})p(z=k|\boldsymbol{\theta},\boldsymbol{\pi}) \\ &= \sum_{k=1}^{K} p(\mathbf{x}|z=k,\boldsymbol{\theta})p(z=k|\boldsymbol{\pi}) = \sum_{k=1}^{K} q(\mathbf{x}|\theta_k)p(z=k|\boldsymbol{\pi}) \end{aligned}$$

Since, by definition,

$$p(\mathbf{x}|\boldsymbol{\theta},\boldsymbol{\pi}) = \sum_{k=1}^{K} \pi_k q(\mathbf{x}_i|\theta_k)$$

it results

$$\pi_k = p(z=k|\boldsymbol{\pi})$$

Responsibilities

An interpretation for $\gamma_k(\mathbf{x})$ can be derived as follows

$$\begin{aligned} \gamma_k(\mathbf{x}) &= \frac{\pi_k q(\mathbf{x}|\theta_k)}{\sum_{j=1}^{K} \pi_j q(\mathbf{x}|\theta_j)} \\ &= \frac{p(z=k)p(\mathbf{x}|z=k)}{\sum_{j=1}^{K} p(z=j)p(\mathbf{x}|z=j)} = p(z=k|\mathbf{x}) \end{aligned}$$

Mixing coefficients and responsibilities

- A mixing coefficient $\pi_k = p(z=k)$ can be seen as the prior (wrt to the observation of the point) probability that the next point is generated by sampling the k-th component distribution
- A responsibility $\gamma_k(\mathbf{x}) = p(z=k|x)$ can be seen as the posterior (wrt to the observation of the point) probability that a point has been generated by sampling the k-th component distribution

In the case, of mixtures of gaussian distribution, we have $q(\mathbf{x}|\theta_k) = \mathcal{N}(\mathbf{x}|\mu_k, \Sigma_k)$. As a consequence,

$$\gamma_k(\mathbf{x}) = \frac{\pi_k \mathcal{N}(\mathbf{x}|\mu_k, \Sigma_k)}{\sum_{j=1}^K \pi_j \mathcal{N}(\mathbf{x}|\mu_j, \Sigma_j)}$$

and the likelihood is maximized for

$$\pi_j = \frac{1}{n}\sum_{i=1}^n \gamma_j(\mathbf{x}_i)$$

$$\sum_{i=1}^n \gamma_j(\mathbf{x}_i)\frac{\partial \log \mathcal{N}(\mathbf{x}_i|\mu_j, \Sigma_j)}{\partial \theta_j} = 0$$

Maximum likelihood

Data set

- Let $\mathrm{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be the set of values of observed variables and let $\boldsymbol{Z} = (z_1, \ldots, z_n)$ be the set of values of the latent variables. Then $(\mathrm{X}, \boldsymbol{Z})$ is the complete dataset: it includes the values of all variables in the model
- X is the observed dataset (incomplete). It only includes “real” data, that is observed data.

Indeed, $\boldsymbol{Z}$ is unknown. If values have been assigned to model parameters, the only possible knowledge about $\boldsymbol{Z}$ is given by the posterior distribution $p(\boldsymbol{Z}|\mathrm{X}, \boldsymbol{\theta}, \boldsymbol{\pi})$.

Inferring parameters for gaussian mixtures

- If we assume that the complete dataset $(\mathrm{X}, \boldsymbol{Z})$ is known (that is the observed points together with their corresponding components) a maximum likelihood estimation of $\boldsymbol{\pi}$ and $\boldsymbol{\theta}$ would be easy. In particular,
- For the mixing coefficients π_k it would result, as usual

  $$\pi_k = \frac{n_k}{n}$$

  where n_k is the number of elements of the set C_k such that $z = k$
- For component parameters $\theta_k = (\mu_k, \Sigma_k)$ the usual estimations for gaussians would provide

  $$\boldsymbol{\mu}_k = \frac{1}{n_k}\sum_{\mathbf{x}\in C_k}\mathbf{x}$$

  $$\Sigma_k = \frac{1}{n_k}\sum_{\mathbf{x}\in C_k}(\mathbf{x} - \boldsymbol{\mu}_k)(\mathbf{x} - \boldsymbol{\mu}_k)^T$$

Log likelihood of complete dataset

The above results derive from the maximimization, wrt π_k, μ_k, Σ_k, $(k = 1, \ldots, K)$ of the log likelihood

$$\begin{aligned} l(\Sigma, \boldsymbol{\mu}, \boldsymbol{\pi}|\mathrm{X}, \boldsymbol{Z}) = \log p(\mathrm{X}, \boldsymbol{Z}|\Sigma, \boldsymbol{\mu}, \boldsymbol{\pi}) &= \log \prod_{i=1}^n \prod_{k=1}^K \pi_k^{\zeta_{ik}} \mathcal{N}(\mathbf{x}_i|\mu_k, \Sigma_k)^{\zeta_{ik}} \\ &= \sum_{i=1}^n \sum_{k=1}^K \zeta_{ik}(\log \pi_k + \log \mathcal{N}(\mathbf{x}_i|\mu_k, \Sigma_k)) \end{aligned}$$

where, ζ_{ik} is the k-component of the 1-to-K coding of z_i, that is, $\zeta_{ik} = 1$ iff $z_i = k$, and 0 otherwise

Dealing with latent variables

Unfortunately, since $\boldsymbol{Z}$ is unknown, the log-likelihood of the complete dataset cannot be defined (the sets C_k are not known).

Our approach will be to consider for maximization, instead of the log-likelihood where each z_i is specified,

- its expectation wrt to the conditional distribution $p(\boldsymbol{Z}|\mathrm{X})$, that is

$$\begin{aligned}E_{p(\boldsymbol{Z}|\mathrm{X})}[l(\Sigma, \boldsymbol{\mu}, \boldsymbol{\pi}|\mathrm{X}, \boldsymbol{Z})] &= \sum_{i=1}^{n}\sum_{k=1}^{K} p(z_i = k|\mathbf{x}_i)(\log \pi_k + \log \mathcal{N}(\mathbf{x}_i|\mu_k, \Sigma_k)) \\ &= \sum_{i=1}^{n}\sum_{k=1}^{K} \gamma_k(\mathbf{x}_i)(\log \pi_k + \log \mathcal{N}(\mathbf{x}_i|\mu_k, \Sigma_k))\end{aligned}$$

Observe that this expectation can be derived if $p(\boldsymbol{Z}|\mathrm{X})$ (that is the set of all values $\gamma_k(\mathbf{x}_i)$) is known.

Maximization of expected log-likelihood

The maximization of $E_{p(\boldsymbol{Z}|\mathrm{X})}[l(\Sigma, \boldsymbol{\mu}, \boldsymbol{\pi}|\mathrm{X}, \boldsymbol{Z})]$ wrt to π_k, μ_k, Σ_k results easily into

$$\begin{aligned}\pi_k &= \frac{1}{n}\sum_{i=1}^{n}\gamma_k(\mathbf{x}_j) \\ \mu_k &= \frac{1}{n_k}\sum_{i=1}^{n}\gamma_k(\mathbf{x}_i)\mathbf{x}_i \\ \Sigma_k &= \frac{1}{n_k}\sum_{i=1}^{n}\gamma_j(\mathbf{x}_i)(\mathbf{x}_i - \boldsymbol{\mu}_k)(\mathbf{x}_i - \boldsymbol{\mu}_k)^T\end{aligned}$$

this is named M-step (from "Maximization")

A new expectation

The computed values for the parameters result into new, different values for $\gamma_k(\mathbf{x}_i) = p(z_i = k|\mathbf{x}_i)$, and a different expectation $E_{p(\boldsymbol{Z}|\mathrm{X})}[l(\Sigma, \boldsymbol{\mu}, \boldsymbol{\pi}|\mathrm{X}, \boldsymbol{Z})]$.

This is named E-step (from "Expectation")

ML and mixtures of gaussians: iterative approach

1. Assign an initial estimate to $\mu_j, \Sigma_j, \pi_j, j = 1, \ldots, K$
2. Repeat
   (a) Compute

$$\gamma_j(x_i) = \frac{1}{\gamma_i}\pi_j\mathcal{N}(x_i|\mu_j, \Sigma_j) \qquad \text{with} \qquad \gamma_i = \sum_{k=1}^{K}\pi_k\mathcal{N}(x_i|\mu_j, \Sigma_j)$$

   (b) Compute

$$\pi_j = \frac{n_j}{n} \qquad \text{with} \qquad n_j = \sum_{i=1}^{n}\gamma_j(x_i)$$

(c) Compute

$$\mu_j = \frac{1}{n_j} \sum_{i=1}^{n} \gamma_j(x_i) x_i$$

(d) Compute

$$\Sigma_j = \frac{1}{n_j} \sum_{i=1}^{n} \gamma_j(x_i)(x_i - \mu_j)(x_i - \mu_j)^T$$

3. until some convergence property is verified

The convergence test may refer to the the increase of log-likelihood in the last iteration

This algorithm is indeed the application of a general schema named **Expectation-Maximization**